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G^1 continuous conditions of biquartic B-spline surfaces [☆]

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Abstract

The necessary and sufficient conditions of G^1 continuity between two biquartic B-spline patches with single interior knots are obtained, and the intrinsic conditions of the common boundary control points are also presented. Further, some remarkable difference on the conditions for G^1 continuity of two adjacent Bézier patches and those of two adjacent B-spline patches are studied. As a result, the construction of local scheme by using biquartic B-spline surfaces with the single interior knots does not exist. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the main task in geometric modelling and CAGD is to reconstruct the surfaces of objects. Parametric surfaces, such as Coons patches, Bézier patches and B-splines patches, are fundamental tools. B-spline surfaces have long become the most popular geometric representation in these fields due to its local interpolation and global approximation properties.

Objects often possess complex surface, so it is very difficult to represent them using a single parametric surface. The common way is to subdivide the surface into such pieces that hold comparatively simple shapes, and to construct a surface patch for each piece. In order to make the resulting surface to be continuous, a widely used method is to interpolate surface patches

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in such a way that neighbor patches meet smoothly. Smoothness can often be satisfied with *geometric continuity*, where only the shape of the adjoining patches is considered, but not their parametrization.

In the past 20 years, the conditions of geometric continuity between two adjacent Bézier patches have been extensively studied in many literatures such as [1–3,7], but little attention has been paid to the conditions for geometric continuity of B-spline surfaces.

In [5], bicubic B-spline surface patches are studied. In this paper, our study will be concentrated on biquartic B-spline surface patches. The main results of this paper include:

- Presenting the G^1 continuous conditions of two adjacent biquartic B-spline surfaces with the single interior knots and the intrinsic conditions of the common boundary control points.
- Introducing the concept of so called *local scheme* [6].
- Showing that the construction of local scheme by using biquartic B-spline surfaces with the single interior knots does not exist.

The paper is organized as follows. In Section 2 we introduce some of the basic concepts and notations that will be utilized repeatedly. And the relations of the control points of defined curves and their piecewise Bézier representation are given. In Section 3, the details of deducing the continuity conditions are discussed. Section 4 contains some discussion.

2. B-spline patches

Suppose $S_1(u, v)$ and $S_2(u, v)$ are two biquartic B-spline surfaces defined over $I^2 = [0, 1] \times [0, 1]$ with the common boundary curve $C(v) = S_1(0, v) = S_2(0, v)$ of the following form:

$$\begin{aligned} S_1(u, v) &= \sum_{i=0}^n \sum_{j=0}^n \mathbf{P}_{i,j} N_{i,4}(u) N_{j,4}(v), \\ S_2(u, v) &= \sum_{i=0}^n \sum_{j=0}^n \mathbf{Q}_{i,j} N_{i,4}(u) N_{j,4}(v) \end{aligned} \quad (1)$$

with the knot vectors

$$U = V = \{0, 0, 0, 0, 0, t_5, t_6, \dots, t_n, 1, 1, 1, 1, 1\},$$

$m := n + 5$. If $n = 4$, then there are no interior knots in U and V , and $S_1(u, v)$ and $S_2(u, v)$ are just two biquartic Bézier patches with the knot vectors $U = V = \{0, 0, 0, 0, 0, 1, 1, 1, 1, 1\}$. Without loss of generality, we may assume the number of control points on the common boundary curve of the two B-spline surfaces is not less than 10 and $h = t_{j+1} - t_j = 1/(n - 3)$ for $j = 4, \dots, n$, but our method used in this paper is suitable for the general cases of U and V .

Now we define three curves

$$\begin{aligned} C_1(v) &= \left. \frac{\partial S_1(u, v)}{\partial u} \right|_{u=0} = \frac{4}{t_5} \sum_{j=0}^n (\mathbf{P}_{1,j} - \mathbf{P}_{0,j}) N_{j,4}(v), \\ C_2(v) &= \left. \frac{\partial S_2(u, v)}{\partial u} \right|_{u=0} = \frac{4}{t_5} \sum_{j=0}^n (\mathbf{Q}_{1,j} - \mathbf{Q}_{0,j}) N_{j,4}(v), \quad v \in [0, 1], \\ C_0(v) &= \left. \frac{\partial S_1(u, v)}{\partial v} \right|_{u=0} = 4 \sum_{j=0}^{n-1} \frac{\mathbf{P}_{0,j+1} - \mathbf{P}_{0,j}}{t_{j+5} - t_{j+1}} N_{j,3}(v), \end{aligned} \quad (2)$$

that is

$$\begin{aligned} C_1(v) &= \frac{4}{h} \sum_{j=0}^n \mathbf{P}_j N_{j,4}(v), \\ C_2(v) &= \frac{4}{h} \sum_{j=0}^n \mathbf{Q}_j N_{j,4}(v), \\ C_0(v) &= 4 \sum_{j=0}^{n-1} \frac{\mathbf{T}_j}{t_{j+5} - t_{j+1}} N_{j,3}(v), \end{aligned} \quad (3)$$

where $\mathbf{P}_j = \mathbf{P}_{1,j} - \mathbf{P}_{0,j}$, $\mathbf{Q}_j = \mathbf{Q}_{1,j} - \mathbf{Q}_{0,j}$ and $\mathbf{T}_j = \mathbf{P}_{0,j+1} - \mathbf{P}_{0,j}$. $C_1(v)$ and $C_2(v)$ are two quartic B-spline curves with the knot vector V ; and $C_0(v)$ is a cubic B-spline curve with the knot vector $V' = \{0, 0, 0, 0, 0, t_5, \dots, t_n, 1, 1, 1, 1\}$.

Inserting the element of $t_5, t_5, t_5, \dots, t_n, t_n, t_n$ into V for the quartic B-spline curve $C_1(v)$ and $C_2(v)$, we have

$$\begin{aligned} C_1(v) &= \frac{4}{h} \sum_{j=0}^{\hat{n}} \hat{\mathbf{P}}_j N_{j,4}(v), \\ C_2(v) &= \frac{4}{h} \sum_{j=0}^{\hat{n}} \hat{\mathbf{Q}}_j N_{j,4}(v), \end{aligned} \quad (4)$$

where $\hat{n} = 4(n-3)$, then $C_1(v)(C_2(v))$ is a quartic B-spline curve with the knot vector

$$\hat{V} = \{0, 0, 0, 0, 0, t_5, t_5, t_5, t_5, \dots, t_n, t_n, t_n, t_n, 1, 1, 1, 1\}.$$

Similarly by inserting the elements of $t_5, t_5, \dots, t_n, t_n$ into V' for $C_0(v)$, we get

$$C_0(v) = \frac{4}{h} \sum_{j=0}^{3(n-3)} \hat{\mathbf{T}}_j N_{j,3}(v). \quad (5)$$

We may use knot insertion formula [4] to compute $\hat{\mathbf{P}}_j$, $\hat{\mathbf{Q}}_j$ and $\hat{\mathbf{T}}_j$, i.e., to decompose $C_i(v)$ ($i = 0, 1, 2$) into its constituent (Bézier) polynomial pieces as follows:

$$\begin{aligned}
 \hat{\mathbf{P}}_0 &= \mathbf{P}_0, \quad \hat{\mathbf{P}}_1 = \mathbf{P}_1, \quad \hat{\mathbf{P}}_2 = \frac{1}{2}(\mathbf{P}_1 + \mathbf{P}_2), \\
 \hat{\mathbf{P}}_3 &= \frac{1}{12}(3\mathbf{P}_1 + 7\mathbf{P}_2 + 2\mathbf{P}_3), \\
 \hat{\mathbf{P}}_4 &= \frac{1}{72}(9\mathbf{P}_1 + 37\mathbf{P}_2 + 23\mathbf{P}_3 + 3\mathbf{P}_4), \\
 \hat{\mathbf{P}}_5 &= \frac{1}{36}(16\mathbf{P}_2 + 17\mathbf{P}_3 + 3\mathbf{P}_4), \\
 \hat{\mathbf{P}}_6 &= \frac{1}{18}(4\mathbf{P}_2 + 11\mathbf{P}_3 + 3\mathbf{P}_4), \\
 \hat{\mathbf{P}}_7 &= \frac{1}{9}(\mathbf{P}_2 + 5\mathbf{P}_3 + 3\mathbf{P}_4), \\
 \hat{\mathbf{P}}_8 &= \frac{1}{72}(4\mathbf{P}_2 + 32\mathbf{P}_3 + 33\mathbf{P}_4 + 3\mathbf{P}_5), \\
 \hat{\mathbf{P}}_{4j} &= \frac{1}{24}(\mathbf{P}_j + 11\mathbf{P}_{j+1} + 11\mathbf{P}_{j+2} + \mathbf{P}_{j+3}), \quad j = 3, \dots, n-6, \\
 \hat{\mathbf{P}}_{4j+1} &= \frac{1}{12}(4\mathbf{P}_{j+1} + 7\mathbf{P}_{j+2} + \mathbf{P}_{j+3}), \quad j = 2, \dots, n-6, \\
 \hat{\mathbf{P}}_{4j+2} &= \frac{1}{6}(\mathbf{P}_{j+1} + 4\mathbf{P}_{j+2} + \mathbf{P}_{j+3}), \quad j = 2, \dots, n-6, \\
 \hat{\mathbf{P}}_{4j+3} &= \frac{1}{12}(\mathbf{P}_{j+1} + 7\mathbf{P}_{j+2} + 4\mathbf{P}_{j+3}), \quad j = 2, \dots, n-6, \\
 \hat{\mathbf{P}}_{4(n-5)} &= \frac{1}{72}(3\mathbf{P}_{n-5} + 33\mathbf{P}_{n-4} + 32\mathbf{P}_{n-3} + 4\mathbf{P}_{n-2}), \\
 \hat{\mathbf{P}}_{4(n-5)+1} &= \frac{1}{9}(3\mathbf{P}_{n-4} + 5\mathbf{P}_{n-3} + \mathbf{P}_{n-2}), \\
 \hat{\mathbf{P}}_{4(n-5)+2} &= \frac{1}{18}(3\mathbf{P}_{n-4} + 11\mathbf{P}_{n-3} + 4\mathbf{P}_{n-2}), \\
 \hat{\mathbf{P}}_{4(n-5)+3} &= \frac{1}{36}(3\mathbf{P}_{n-4} + 17\mathbf{P}_{n-3} + 16\mathbf{P}_{n-2}), \\
 \hat{\mathbf{P}}_{4(n-4)} &= \frac{1}{72}(3\mathbf{P}_{n-4} + 23\mathbf{P}_{n-3} + 37\mathbf{P}_{n-2} + 9\mathbf{P}_{n-1}), \\
 \hat{\mathbf{P}}_{4(n-4)+1} &= \frac{1}{12}(2\mathbf{P}_{n-3} + 7\mathbf{P}_{n-2} + 3\mathbf{P}_{n-1}), \\
 \hat{\mathbf{P}}_{4(n-4)+2} &= \frac{1}{2}(\mathbf{P}_{n-2} + \mathbf{P}_{n-1}), \\
 \hat{\mathbf{P}}_{4(n-4)+3} &= \mathbf{P}_{n-1}, \quad \hat{\mathbf{P}}_{4(n-3)} = \mathbf{P}_n
 \end{aligned} \tag{6}$$

and

$$\begin{aligned}
 \hat{\mathbf{T}}_0 &= \mathbf{T}_0, \quad \hat{\mathbf{T}}_1 = \frac{1}{2}\mathbf{T}_1, \quad \hat{\mathbf{T}}_2 = \frac{1}{12}(3\mathbf{T}_1 + 2\mathbf{T}_2), \\
 \hat{\mathbf{T}}_3 &= \frac{1}{72}(9\mathbf{T}_1 + 14\mathbf{T}_2 + 3\mathbf{T}_3),
 \end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{T}}_4 &= \frac{1}{36}(8\mathbf{T}_2 + 3\mathbf{T}_3), \quad \hat{\mathbf{T}}_5 = \frac{1}{18}(2\mathbf{T}_2 + 3\mathbf{T}_3), \\
\hat{\mathbf{T}}_6 &= \frac{1}{72}(4\mathbf{T}_2 + 12\mathbf{T}_3 + 3\mathbf{T}_4), \\
\hat{\mathbf{T}}_{3j} &= \frac{1}{24}(\mathbf{T}_j + 4\mathbf{T}_{j+1} + \mathbf{T}_{j+2}), \quad j = 3, \dots, n-6, \\
\hat{\mathbf{T}}_{3j+1} &= \frac{1}{12}(2\mathbf{T}_{j+1} + \mathbf{T}_{j+2}), \quad j = 2, \dots, n-6, \\
\hat{\mathbf{T}}_{3j+2} &= \frac{1}{12}(\mathbf{T}_{j+1} + 2\mathbf{T}_{j+2}), \quad j = 2, \dots, n-6, \\
\hat{\mathbf{T}}_{3(n-5)} &= \frac{1}{72}(3\mathbf{T}_{n-5} + 12\mathbf{T}_{n-4} + 4\mathbf{T}_{n-3}), \\
\hat{\mathbf{T}}_{3(n-5)+1} &= \frac{1}{18}(3\mathbf{T}_{n-4} + 2\mathbf{T}_{n-3}), \\
\hat{\mathbf{T}}_{3(n-5)+2} &= \frac{1}{36}(3\mathbf{T}_{n-4} + 8\mathbf{T}_{n-3}), \\
\hat{\mathbf{T}}_{3(n-4)} &= \frac{1}{72}(3\mathbf{T}_{n-4} + 14\mathbf{T}_{n-3} + 9\mathbf{T}_{n-2}), \\
\hat{\mathbf{T}}_{3(n-4)+1} &= \frac{1}{12}(2\mathbf{T}_{n-3} + 3\mathbf{T}_{n-2}), \\
\hat{\mathbf{T}}_{3(n-4)+2} &= \frac{1}{2}\mathbf{T}_{n-1}, \quad \hat{\mathbf{T}}_{3(n-3)} = \mathbf{T}_{n-1}.
\end{aligned} \tag{7}$$

$\hat{\mathbf{Q}}_j$ are similarly defined as $\hat{\mathbf{P}}_j$ in (6).

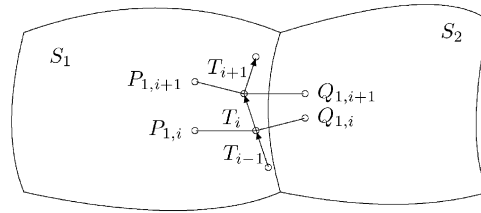
In this paper, we always assume that $n \geq 10$ for simplification, other cases can be discussed similarly.

3. G^1 continuous conditions of two adjacent B-spline patches

The restrictions of $C_1(v)$, $C_2(v)$ and $C_0(v)$ on the interval $[t_{j+4}, t_{j+5}]$, $j = 0, 1, \dots, n-4$ with $t_4 = 0$, $t_{n+1} = 1$ are, respectively,

$$\begin{aligned}
C_{1,j}(t) &:= \sum_{i=0}^4 \hat{\mathbf{P}}_{4j+i} B_{i,4}(\hat{t}), \\
C_{2,j}(t) &:= \sum_{i=0}^4 \hat{\mathbf{Q}}_{4j+i} B_{i,4}(\hat{t}), \\
C_{0,j}(t) &:= \sum_{i=0}^3 \hat{\mathbf{T}}_{3j+i} B_{i,3}(\hat{t}),
\end{aligned}$$

where $\hat{t} = (t - t_j)/(t_{j+1} - t_j)$ and $B_{i,p}(\hat{t}) = C_p^i \hat{t}^i (1 - \hat{t})^{p-i}$.

Fig. 1. G^1 Continuity of two adjacent B-spline patches.

Then $C_{1,j}$, $C_{2,j}$ and $C_{0,j}$ for $j=0, \dots, n-4$ are G^1 smooth joint if there exist three functions $h_j(\hat{t})$, $f_j(\hat{t})$ and $g_j(\hat{t})$ such that

$$h_j(\hat{t}) \sum_{i=0}^4 \hat{\mathbf{Q}}_{4j+i} B_{i,4}(\hat{t}) = f_j(\hat{t}) \sum_{i=0}^4 \hat{\mathbf{P}}_{4j+i} B_{i,4}(\hat{t}) + g_j(\hat{t}) \sum_{i=0}^3 \hat{\mathbf{T}}_{3j+i} B_{i,3}(\hat{t}), \quad (8)$$

where $h_j(\hat{t})f_j(\hat{t}) < 0$. In almost all existing literatures related to constructing G^1 surface models, it is usual to take

$$\begin{aligned} h_j(\hat{t}) &= 1, \\ f_j(\hat{t}) &= -1, \\ g_j(\hat{t}) &= b_j(1 - \hat{t}) + c_j\hat{t}. \end{aligned} \quad (9)$$

So we use (9) to yield

$$\begin{aligned} \hat{\mathbf{Q}}_{4j} &= -\hat{\mathbf{P}}_{4j} + b_j\hat{\mathbf{T}}_{3j}, \\ 4\hat{\mathbf{Q}}_{4j+1} &= -4\hat{\mathbf{P}}_{4j+1} + 3b_j\hat{\mathbf{T}}_{3j+1} + c_j\hat{\mathbf{T}}_{3j}, \\ 2\hat{\mathbf{Q}}_{4j+2} &= -2\hat{\mathbf{P}}_{4j+2} + b_j\hat{\mathbf{T}}_{3j+2} + c_j\hat{\mathbf{T}}_{3j+1}, \quad j=0, \dots, n-4, \\ 4\hat{\mathbf{Q}}_{4j+3} &= -4\hat{\mathbf{P}}_{4j+3} + b_j\hat{\mathbf{T}}_{3j+3} + 3c_j\hat{\mathbf{T}}_{3j+2}, \\ \hat{\mathbf{Q}}_{4(j+1)} &= -\hat{\mathbf{P}}_{4(j+1)} + c_j\hat{\mathbf{T}}_{3j+3}. \end{aligned} \quad (10)$$

Eq. (10) is a set of G^1 conditions between two Bézier patches (see Fig. 1).

From the first and the last equations of (10), we get

$$b_{j+1} = c_j, \quad j=0, \dots, n-5. \quad (11)$$

Rewrite (10) as the following forms:

$$\begin{aligned} \hat{\mathbf{Q}}_{4j} &= -\hat{\mathbf{P}}_{4j} + b_j\hat{\mathbf{T}}_{3j}, \\ 4\hat{\mathbf{Q}}_{4j+1} &= -4\hat{\mathbf{P}}_{4j+1} + 3b_j\hat{\mathbf{T}}_{3j+1} + c_j\hat{\mathbf{T}}_{3j}, \end{aligned}$$

$$\begin{aligned}
2\hat{\mathbf{Q}}_{4j+2} &= -2\hat{\mathbf{P}}_{4j+2} + b_j\hat{\mathbf{T}}_{3j+2} + c_j\hat{\mathbf{T}}_{3j+1}, \quad j=0, \dots, n-5, \\
4\hat{\mathbf{Q}}_{4j+3} &= -4\hat{\mathbf{P}}_{4j+3} + b_j\hat{\mathbf{T}}_{3j+3} + 3c_j\hat{\mathbf{T}}_{3j+2}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\hat{\mathbf{Q}}_{4(n-4)} &= -\hat{\mathbf{P}}_{4(n-4)} + b_{n-4}\hat{\mathbf{T}}_{3(n-4)}, \\
4\hat{\mathbf{Q}}_{4(n-4)+1} &= -4\hat{\mathbf{P}}_{4(n-4)+1} + 3b_{n-4}\hat{\mathbf{T}}_{3(n-4)+1} + c_{n-4}\hat{\mathbf{T}}_{3(n-4)}, \\
2\hat{\mathbf{Q}}_{4(n-4)+2} &= -2\hat{\mathbf{P}}_{4(n-4)+2} + b_{n-4}\hat{\mathbf{T}}_{3(n-4)+2} + c_{n-4}\hat{\mathbf{T}}_{3(n-4)+1}, \\
4\hat{\mathbf{Q}}_{4(n-4)+3} &= -4\hat{\mathbf{P}}_{4(n-4)+3} + b_{n-4}\hat{\mathbf{T}}_{3(n-4)+3} + 3c_{n-4}\hat{\mathbf{T}}_{3(n-4)+2}, \\
\hat{\mathbf{Q}}_{4(n-3)} &= -\hat{\mathbf{P}}_{4(n-3)} + c_{n-4}\hat{\mathbf{T}}_{3(n-3)}.
\end{aligned} \tag{13}$$

For convenience, we denote

$$\mathbf{M}_j = \mathbf{Q}_j + \mathbf{P}_j, \quad j=0, \dots, n. \tag{14}$$

Substituting (6) and (7) into (12) and (13), we obtain

$$\begin{aligned}
\mathbf{M}_0 &= b_0\mathbf{T}_0, \\
4\mathbf{M}_1 &= \frac{3}{2}b_0\mathbf{T}_1 + c_0\mathbf{T}_0, \\
4(\mathbf{M}_1 + \mathbf{M}_2) &= \frac{b_0}{3}(3\mathbf{T}_1 + 2\mathbf{T}_2) + 2c_0\mathbf{T}_1, \\
4(3\mathbf{M}_1 + 7\mathbf{M}_2 + 2\mathbf{M}_3) &= \frac{b_0}{6}(9\mathbf{T}_1 + 14\mathbf{T}_2 + 3\mathbf{T}_3) + 3c_0(3\mathbf{T}_1 + 2\mathbf{T}_2),
\end{aligned} \tag{15}$$

$$\begin{aligned}
9\mathbf{M}_1 + 37\mathbf{M}_2 + 23\mathbf{M}_3 + 3\mathbf{M}_4 &= c_0(9\mathbf{T}_1 + 14\mathbf{T}_2 + 3\mathbf{T}_3), \\
4(16\mathbf{M}_2 + 17\mathbf{M}_3 + 3\mathbf{M}_4) &= 3c_0(8\mathbf{T}_2 + 3\mathbf{T}_3) + \frac{c_1}{2}(9\mathbf{T}_1 + 14\mathbf{T}_2 + 3\mathbf{T}_3), \\
4(4\mathbf{M}_2 + 11\mathbf{M}_3 + 3\mathbf{M}_4) &= 2c_0(2\mathbf{T}_2 + 3\mathbf{T}_3) + c_1(8\mathbf{T}_2 + 3\mathbf{T}_3), \\
4(\mathbf{M}_2 + 5\mathbf{M}_3 + 3\mathbf{M}_4) &= \frac{c_0}{8}(4\mathbf{T}_2 + 12\mathbf{T}_3 + 3\mathbf{T}_4) + \frac{3}{2}c_1(2\mathbf{T}_2 + 3\mathbf{T}_3),
\end{aligned} \tag{16}$$

$$\begin{aligned}
4\mathbf{M}_2 + 32\mathbf{M}_3 + 33\mathbf{M}_4 + 3\mathbf{M}_5 &= c_1(4\mathbf{T}_2 + 12\mathbf{T}_3 + 3\mathbf{T}_4), \\
4(4\mathbf{M}_3 + 7\mathbf{M}_4 + \mathbf{M}_5) &= 3c_1(2\mathbf{T}_3 + \mathbf{T}_4) + \frac{c_2}{6}(4\mathbf{T}_2 + 12\mathbf{T}_3 + 3\mathbf{T}_4), \\
4(\mathbf{M}_3 + 4\mathbf{M}_4 + \mathbf{M}_5) &= c_1(\mathbf{T}_3 + 2\mathbf{T}_4) + c_2(2\mathbf{T}_3 + \mathbf{T}_4), \\
4(\mathbf{M}_3 + 7\mathbf{M}_4 + 4\mathbf{M}_5) &= \frac{c_1}{2}(\mathbf{T}_3 + 4\mathbf{T}_4 + \mathbf{T}_5) + 3c_2(\mathbf{T}_3 + 2\mathbf{T}_4),
\end{aligned} \tag{17}$$

$$\begin{aligned}
\mathbf{M}_j + 11\mathbf{M}_{j+1} + 11\mathbf{M}_{j+2} + \mathbf{M}_{j+3} &= c_{j-1}(\mathbf{T}_j + 4\mathbf{T}_{j+1} + \mathbf{T}_{j+2}), \\
4(4\mathbf{M}_{j+1} + 7\mathbf{M}_{j+2} + \mathbf{M}_{j+3}) &= 3c_{j-1}(2\mathbf{T}_{j+1} + \mathbf{T}_{j+2}) + \frac{c_j}{2}(\mathbf{T}_j + 4\mathbf{T}_{j+1} + \mathbf{T}_{j+2}), \\
4(\mathbf{M}_{j+1} + 4\mathbf{M}_{j+2} + \mathbf{M}_{j+3}) &= c_{j-1}(\mathbf{T}_{j+1} + 2\mathbf{T}_{j+2}) + c_j(2\mathbf{T}_{j+1} + \mathbf{T}_{j+2}), \\
4(\mathbf{M}_{j+1} + 7\mathbf{M}_{j+2} + 4\mathbf{M}_{j+3}) &= \frac{c_{j-1}}{2}(\mathbf{T}_{j+1} + 4\mathbf{T}_{j+2} + \mathbf{T}_{j+3}) + 3c_j(\mathbf{T}_{j+1} + 2\mathbf{T}_{j+2}), \\
j &= 3, \dots, n-7,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathbf{M}_{n-6} + 11\mathbf{M}_{n-5} + 11\mathbf{M}_{n-4} + \mathbf{M}_{n-3} &= c_{n-7}(\mathbf{T}_{n-6} + 4\mathbf{T}_{n-5} + \mathbf{T}_{n-4}), \\
4(4\mathbf{M}_{n-5} + 7\mathbf{M}_{n-4} + \mathbf{M}_{n-3}) &= 3c_{n-7}(2\mathbf{T}_{n-5} + \mathbf{T}_{n-4}) + \frac{c_{n-6}}{2}(\mathbf{T}_{n-6} + 4\mathbf{T}_{n-5} + \mathbf{T}_{n-4}), \\
4(\mathbf{M}_{n-5} + 4\mathbf{M}_{n-4} + \mathbf{M}_{n-3}) &= c_{n-7}(\mathbf{T}_{n-5} + 2\mathbf{T}_{n-4}) + c_{n-6}(2\mathbf{T}_{n-5} + \mathbf{T}_{n-4}), \\
4(\mathbf{M}_{n-5} + 7\mathbf{M}_{n-4} + 4\mathbf{M}_{n-3}) &= \frac{c_{n-7}}{6}(3\mathbf{T}_{n-5} + 12\mathbf{T}_{n-4} + 4\mathbf{T}_{n-3}) + 3c_{n-6}(\mathbf{T}_{n-5} + 2\mathbf{T}_{n-4}), \\
\end{aligned} \tag{19}$$

$$\begin{aligned}
3\mathbf{M}_{n-5} + 33\mathbf{M}_{n-4} + 32\mathbf{M}_{n-3} + 4\mathbf{M}_{n-2} &= c_{n-6}(3\mathbf{T}_{n-5} + 12\mathbf{T}_{n-4} + 4\mathbf{T}_{n-3}), \\
4(3\mathbf{M}_{n-4} + 5\mathbf{M}_{n-3} + \mathbf{M}_{n-2}) &= \frac{3}{2}c_{n-6}(3\mathbf{T}_{n-4} + 2\mathbf{T}_{n-3}) + \frac{c_{n-5}}{8}(3\mathbf{T}_{n-5} + 12\mathbf{T}_{n-4} + 4\mathbf{T}_{n-3}), \\
4(3\mathbf{M}_{n-4} + 11\mathbf{M}_{n-3} + 4\mathbf{M}_{n-2}) &= c_{n-6}(3\mathbf{T}_{n-4} + 8\mathbf{T}_{n-3}) + 2c_{n-5}(3\mathbf{T}_{n-4} + 2\mathbf{T}_{n-3}), \\
4(3\mathbf{M}_{n-4} + 17\mathbf{M}_{n-3} + 16\mathbf{M}_{n-2}) &= \frac{c_{n-6}}{2}(3\mathbf{T}_{n-4} + 14\mathbf{T}_{n-3} + 9\mathbf{T}_{n-2}) + 3c_{n-5}(3\mathbf{T}_{n-4} + 8\mathbf{T}_{n-3}), \\
\end{aligned} \tag{20}$$

$$\begin{aligned}
3\mathbf{M}_{n-4} + 23\mathbf{M}_{n-3} + 37\mathbf{M}_{n-2} + 9\mathbf{M}_{n-1} &= c_{n-5}(3\mathbf{T}_{n-4} + 14\mathbf{T}_{n-3} + 9\mathbf{T}_{n-2}), \\
4(2\mathbf{M}_{n-3} + 7\mathbf{M}_{n-2} + 3\mathbf{M}_{n-1}) &= 3c_{n-5}(2\mathbf{T}_{n-3} + 3\mathbf{T}_{n-2}) + \frac{c_{n-4}}{6}(3\mathbf{T}_{n-4} + 14\mathbf{T}_{n-3} + 9\mathbf{T}_{n-2}), \\
4(\mathbf{M}_{n-2} + \mathbf{M}_{n-1}) &= 2c_{n-5}\mathbf{T}_{n-2} + \frac{c_{n-4}}{3}(2\mathbf{T}_{n-3} + 3\mathbf{T}_{n-2}), \\
4\mathbf{M}_{n-1} &= c_{n-5}\mathbf{T}_{n-1} + \frac{3}{2}c_{n-4}\mathbf{T}_{n-2}, \\
\mathbf{M}_n &= c_{n-4}\mathbf{T}_{n-1}.
\end{aligned} \tag{21}$$

From the last equations of (15) and the first two equations of (16), we obtain

$$b_0 + c_1 = 2c_0. \tag{22}$$

By the same deduction as obtaining (22), we get

$$\begin{aligned}
b_0 + c_1 &= 2c_0, \\
c_j + c_{j+2} &= 2c_{j+1}, \quad j = 0, \dots, n-6
\end{aligned} \tag{23}$$

and the first equation of (16)–(21) can be deleted from their groups and have no effect to the result. We solve \mathbf{M}_j ($j=0, \dots, n$) from Eqs. (15)–(21) to obtain

$$\begin{aligned}
 \mathbf{M}_0 &= -b_0 \mathbf{T}_0, \\
 \mathbf{M}_1 &= \frac{3}{8}b_0 \mathbf{T}_1 + \frac{c_0}{4} \mathbf{T}_0, \\
 \mathbf{M}_2 &= \frac{1}{8}c_1 \mathbf{T}_1 + \left(\frac{11}{24}c_0 - \frac{1}{6}c_1 \right) \mathbf{T}_2 - \frac{1}{24}c_0 \mathbf{T}_3 + \frac{c_0}{96} \mathbf{T}_4, \\
 \mathbf{M}_3 &= -\frac{1}{16}c_1 \mathbf{T}_1 + \left(\frac{7}{24}c_1 - \frac{c_0}{12} \right) \mathbf{T}_2 + \left(\frac{5}{24}c_0 - \frac{c_1}{16} \right) \mathbf{T}_3 - \frac{c_0}{48} \mathbf{T}_4, \\
 \mathbf{M}_4 &= -\frac{c_2}{36} \mathbf{T}_2 + \left(\frac{5}{24}c_2 - \frac{c_1}{16} \right) \mathbf{T}_3 + \left(\frac{5}{24}c_1 - \frac{c_2}{16} \right) \mathbf{T}_4 - \frac{c_1}{48} \mathbf{T}_5, \\
 \mathbf{M}_{j+2} &= -\frac{c_j}{48} \mathbf{T}_j + \left(\frac{5}{24}c_j - \frac{c_{j-1}}{16} \right) \mathbf{T}_{j+1} + \left(\frac{5}{24}c_{j-1} - \frac{c_j}{16} \right) \mathbf{T}_{j+2} - \frac{c_{j-1}}{48} \mathbf{T}_{j+3}, \\
 j &= 3, \dots, n-7, \\
 \mathbf{M}_{n-4} &= -\frac{c_{n-6}}{48} \mathbf{T}_{n-6} + \left(\frac{5}{24}c_{n-6} - \frac{c_{n-7}}{16} \right) \mathbf{T}_{n-5} + \left(\frac{5}{24}c_{n-7} - \frac{c_{n-6}}{16} \right) \mathbf{T}_{n-4} - \frac{c_{n-7}}{36} \mathbf{T}_{n-3}, \\
 \mathbf{M}_{n-3} &= -\frac{c_{n-5}}{48} \mathbf{T}_{n-5} + \left(\frac{5}{24}c_{n-5} - \frac{c_{n-6}}{16} \right) \mathbf{T}_{n-4} + \left(\frac{7}{24}c_{n-6} - \frac{c_{n-5}}{12} \right) \mathbf{T}_{n-3} - \frac{c_{n-6}}{16} \mathbf{T}_{n-2}, \\
 \mathbf{M}_{n-2} &= \frac{c_{n-4}}{6} \mathbf{T}_{n-3} + \left(\frac{c_{n-5}}{2} - \frac{c_{n-4}}{8} \right) \mathbf{T}_{n-2} - \frac{c_{n-5}}{4} \mathbf{T}_{n-1}, \\
 \mathbf{M}_{n-1} &= \frac{3}{8}c_{n-4} \mathbf{T}_{n-2} + \frac{c_{n-5}}{4} \mathbf{T}_{n-1}, \\
 \mathbf{M}_n &= c_{n-4} \mathbf{T}_{n-1}
 \end{aligned} \tag{24}$$

and

$$\begin{aligned}
 c_0(24\mathbf{T}_0 - 24\mathbf{T}_1 + 12\mathbf{T}_2 - 4\mathbf{T}_3 + \mathbf{T}_4) &= 0, \\
 c_1(3\mathbf{T}_1 - 6\mathbf{T}_2 + 6\mathbf{T}_3 - 4\mathbf{T}_4 + \mathbf{T}_5) &= 0, \\
 c_2\left(\frac{4}{3}\mathbf{T}_2 - 4\mathbf{T}_3 + 6\mathbf{T}_4 - 4\mathbf{T}_5 + \mathbf{T}_6\right) &= 0, \\
 c_j(\mathbf{T}_j - 4\mathbf{T}_{j+1} + 6\mathbf{T}_{j+2} - 4\mathbf{T}_{j+3} + \mathbf{T}_{j+4}) &= 0, \quad j=3, \dots, n-8, \\
 c_{n-7}(\mathbf{T}_{n-7} - 4\mathbf{T}_{n-6} + 6\mathbf{T}_{n-5} - 4\mathbf{T}_{n-4} + \frac{4}{3}\mathbf{T}_{n-3}) &= 0, \\
 c_{n-6}(\mathbf{T}_{n-6} - 4\mathbf{T}_{n-5} + 6\mathbf{T}_{n-4} - 4\mathbf{T}_{n-3} + 3\mathbf{T}_{n-2}) &= 0, \\
 c_{n-5}(\mathbf{T}_{n-5} - 4\mathbf{T}_{n-4} + 12\mathbf{T}_{n-3} - 24\mathbf{T}_{n-2} + 24\mathbf{T}_{n-1}) &= 0.
 \end{aligned}$$

If $c_0 c_1 \cdots c_{n-5} \neq 0$, the above equations are

$$\begin{aligned}
 24\mathbf{T}_0 - 24\mathbf{T}_1 + 12\mathbf{T}_2 - 4\mathbf{T}_3 + \mathbf{T}_4 &= 0, \\
 3\mathbf{T}_1 - 6\mathbf{T}_2 + 6\mathbf{T}_3 - 4\mathbf{T}_4 + \mathbf{T}_5 &= 0, \\
 \frac{4}{3}\mathbf{T}_2 - 4\mathbf{T}_3 + 6\mathbf{T}_4 - 4\mathbf{T}_5 + \mathbf{T}_6 &= 0, \\
 \mathbf{T}_j - 4\mathbf{T}_{j+1} + 6\mathbf{T}_{j+2} - 4\mathbf{T}_{j+3} + \mathbf{T}_{j+4} &= 0, \quad j=3, \dots, n-8, \\
 \mathbf{T}_{n-7} - 4\mathbf{T}_{n-6} + 6\mathbf{T}_{n-5} - 4\mathbf{T}_{n-4} + \frac{4}{3}\mathbf{T}_{n-3} &= 0, \\
 \mathbf{T}_{n-6} - 4\mathbf{T}_{n-5} + 6\mathbf{T}_{n-4} - 4\mathbf{T}_{n-3} + 3\mathbf{T}_{n-2} &= 0, \\
 \mathbf{T}_{n-5} - 4\mathbf{T}_{n-4} + 12\mathbf{T}_{n-3} - 24\mathbf{T}_{n-2} + 24\mathbf{T}_{n-1} &= 0.
 \end{aligned} \tag{25}$$

Eqs. (25) are called intrinsic equations of G^1 about the smoothness condition (8).

Theorem 1. *For the G^1 condition (8), the boundary control vectors of two G^1 adjacent biquartic B-spline patches have to satisfy Eqs. (25) in case $c_i \neq 0$, $i=0, 1, \dots, n-5$.*

Solving (23) to obtain

$$c_i = b_0 + \frac{i+1}{n-3}(c_{n-4} - b_0), \quad i=0, \dots, n-4. \tag{26}$$

Therefore, $c_i = 0$, $i=0, 1, \dots, n-4$, corresponding to the simple collinear G^1 condition. This case has been heavily studied, so we will not consider this case here. In the following, we solve Eqs. (25) for \mathbf{T}_j 's, the detail is below.

From the fourth equation of Eqs. (25), we have

$$\Delta^3 \mathbf{T}_{j+1} = \Delta^3 \mathbf{T}_j, \quad j=3, \dots, n-8, \tag{27}$$

where $\Delta \mathbf{T}_j = \mathbf{T}_{j+1} - \mathbf{T}_j$ and $\Delta^2 \mathbf{T}_j = \Delta \mathbf{T}_{j+1} - \Delta \mathbf{T}_j$ and $\Delta^3 = \Delta(\Delta^2)$.

From the third equation of (25), there holds

$$\Delta^3 \mathbf{T}_3 = \Delta^3 \mathbf{T}_2 + \frac{1}{3} \mathbf{T}_2. \tag{28}$$

From (27) and (28), there holds

$$\Delta^3 \mathbf{T}_j = \Delta^3 \mathbf{T}_2 + \frac{1}{3} \mathbf{T}_2, \quad j=3, \dots, n-7. \tag{29}$$

From the fifth equation of (25)

$$\Delta^3 \mathbf{T}_{n-6} = \Delta^3 \mathbf{T}_3 + \frac{1}{3}(\mathbf{T}_2 - \mathbf{T}_{n-3}). \tag{30}$$

By the above equation, we obtain

$$\begin{aligned} \mathbf{T}_{j+5} &= \mathbf{T}_5 + j\Delta\mathbf{T}_4 + \frac{j(j+1)}{2}\Delta^2\mathbf{T}_3 + \frac{j(j+1)(j+2)}{6}\left(\Delta^3\mathbf{T}_2 + \frac{1}{3}\mathbf{T}_2\right), \quad j = 1, \dots, n-9, \\ \mathbf{T}_{n-3} &= \frac{3}{4}\left[\mathbf{T}_5 + (n-8)\Delta\mathbf{T}_4 + \frac{(n-8)(n-7)}{2}\Delta^2\mathbf{T}_3 \right. \\ &\quad \left. + \frac{(n-8)(n-7)(n-6)}{6}\left(\Delta^3\mathbf{T}_2 + \frac{1}{3}\mathbf{T}_2\right)\right]. \end{aligned} \quad (31)$$

From the first and second equations of (25), there holds

$$\begin{aligned} \mathbf{T}_5 &= -96\mathbf{T}_0 + 93\mathbf{T}_1 - 42\mathbf{T}_2 + 10\mathbf{T}_3, \\ \Delta\mathbf{T}_4 &= -72\mathbf{T}_0 + 69\mathbf{T}_1 - 30\mathbf{T}_2 + 6\mathbf{T}_3, \\ \Delta^2\mathbf{T}_3 &= -48\mathbf{T}_0 + 45\mathbf{T}_1 - 18\mathbf{T}_2 + 3\mathbf{T}_3, \\ \Delta^3\mathbf{T}_2 &= -24\mathbf{T}_0 + 21\mathbf{T}_1 - 5\mathbf{T}_2 + \mathbf{T}_3. \end{aligned} \quad (32)$$

From the definition of \mathbf{T}_j , we obtain

$$\mathbf{T}_0 + \mathbf{T}_1 + \dots + \mathbf{T}_{n-1} = \mathbf{P}_{0,n} - \mathbf{P}_{0,0}. \quad (33)$$

Substituting (31) and (32) into the last two equations of (25) and (33), we find that $\mathbf{T}_2, \dots, \mathbf{T}_{n-2}$ can be expressed by $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_{n-1}$.

Theorem 2. For the intrinsic equations (25), $\mathbf{T}_2, \dots, \mathbf{T}_{n-2}$ can be determined by $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_{n-1}$.

4. Discussion

It is easy to see that the intrinsic equations are the special phenomena of B-spline surfaces, and do not exist in Bernstein–Bézier patches. Another fact deserving our attention is, according to Theorem 2, when we construct G^1 surface models using biquartic B-spline surface with single interior knots, we cannot acquire more freedom by increasing the number of control points.

Considering the complexity of G^1 continuity around the corner \mathbf{P} , we adopt the following three steps to solve it (refer to Fig. 2):

1. Around the corner \mathbf{P} , determine the corner control points $\mathbf{A}_i, \mathbf{B}_i, \mathbf{I}_i$.
2. Along each boundary, determine the remain control points appearing in (10).
3. Determine other control points.

The scheme constructed by the above three steps is called *local scheme*.

Therefore, the first and the last three control vectors of each boundary Γ_i have to be chosen freely in order to construct a local scheme, i.e., $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_{n-2}, \mathbf{T}_{n-1}$ in (25) have to be chosen freely. This is contrary to Theorem 2. Hence, a local scheme of constructing G^1 smooth surface models by using biquartic B-splines with the single interior knots does not exist.

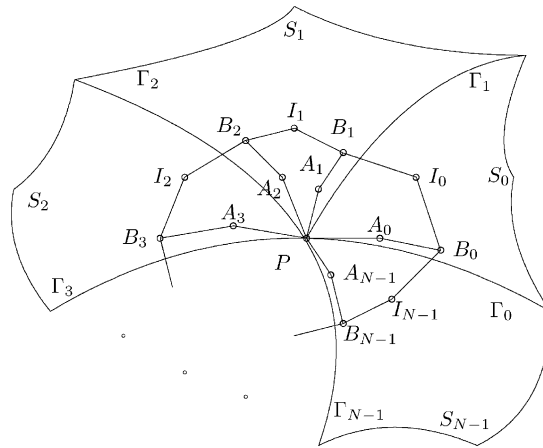


Fig. 2. G^1 continuity of N -patch meeting at a common corner.

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